

② Directional derivative; Gradient ()

$T(x, y, z) \rightarrow$ Temperature

Let us know T at every point of a room or a metal bar,

\Rightarrow rate of change of T with distance changes

(\uparrow or \downarrow in some direction)

\rightarrow Rate of change of T depends on direction in which we move; consequently, it is called directional derivative.

\Rightarrow We want to find

$$\frac{\Delta T}{\Delta s}$$

$\Delta s \leftarrow$ element of distance
(arc length)

in corresponding distance

We write directional derivative as $\frac{dT}{ds}$

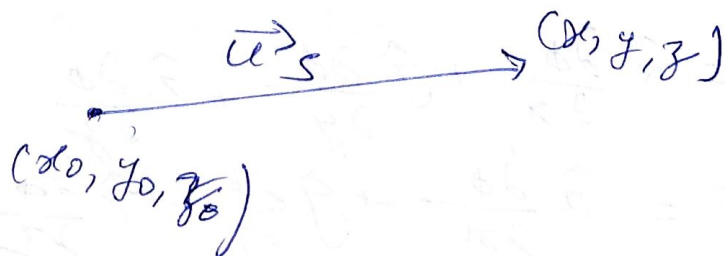
\rightarrow heat flow from large $\frac{dT}{ds}$ to low $\frac{dT}{ds}$

Take scalar function $\phi(x, y, z)$

We want to calculate $\frac{d\phi}{ds}$

\rightarrow rate of change of ϕ with distance at a given point (x_0, y_0, z_0) and in a given direction

\rightarrow let $\vec{u} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c} \rightarrow$ unit vector in given direction



We start from (x_0, y_0, z_0) and go a distance 's' in direction of \vec{u} to point (x, y, z)

\Downarrow
Vector joining these two points is $\vec{u}s$.

$\left. \begin{array}{l} \text{if } \vec{u} \text{ is unit} \\ \text{vector} \end{array} \right\}$

$$\therefore (x, y, z) - (x_0, y_0, z_0) = \vec{u}s = (a\hat{i} + b\hat{j} + c\hat{k})s$$

$$\Rightarrow \left. \begin{array}{l} x = x_0 + as \\ y = y_0 + bs \\ z = z_0 + cs \end{array} \right\} \text{--- (1)}$$

$x, y, z \rightarrow$ functions of s only (single variable)

Now ϕ becomes function of one variable.

$$\text{Now } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial z} \frac{dz}{ds} \text{--- (2)}$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} a + \frac{\partial \phi}{\partial y} b + \frac{\partial \phi}{\partial z} c \text{--- (3)}$$

We see that (3) is dot product of \vec{u} with vector $i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$

We call this vector as grad ϕ or $\nabla \phi$

$$\nabla \phi = \text{grad } \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \text{--- (4)}$$

$$\frac{d\phi}{ds} = \nabla \phi \cdot \vec{u} \rightarrow \text{directional derivative} \text{--- (5)}$$