② Directional derivative; Gradient ( )

$T(x,y,z) \rightarrow$ Temperature

Let we know T at every point of a room or a metal bar,

$\Rightarrow$ rate of change of T with distance changes

(ϕ^s or t_s in some direction)

$\rightarrow$ Rate of change of T depends on direction in which we move; consequently it is called directional derivative

$\Rightarrow$ We want to find $\frac{\Delta T}{\Delta S}$ ← element of distance (arc length) in corresponding distance

We write directional derivative as $\frac{dT}{ds}$

$\rightarrow$ heat flow from large $\frac{dT}{ds}$ to low $\frac{dT}{ds}$

Take scalar function $\phi(x,y,z)$

We want to calculate $\frac{d\phi}{ds}$

$\rightarrow$ rate of change of ϕ with distance at a given point (x_0, y_0, z_0) and in a given direction

$\rightarrow$ let $\vec{u} = ia + jb + kc$ $\rightarrow$ unit vector in given direction

$$(x_0, y_0, z_0) \xrightarrow{\vec{u}\; s} (x, y, z)$$

We start from (x_0, y_0, z_0) and go a distance 's' in direction of $\vec{u}$ to point (x, y, z)

$\Downarrow$

vector joining these two point is $\vec{u}s$.

{$\because \vec{u}$ is unit vector}

$$\therefore (x, y, z) - (x_0, y_0, z_0) = \vec{u}s = (a\hat{i} + b\hat{j} + c\hat{k})s$$

$$\Rightarrow \left.\begin{aligned} x &= x_0 + as \\ y &= y_0 + bs \\ z &= z_0 + cs \end{aligned}\right\} \quad \text{—— (1)}$$

$x, y, z \rightarrow$ functions of s only (single variable)

Now ϕ becomes function of one variables.

Now $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} \quad \text{—— (2)}$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} a + \frac{\partial \phi}{\partial y} b + \frac{\partial \phi}{\partial z} c \quad \text{—— (3)}$$

We see that (3) is dot product of $\vec{u}$ with vector $i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

We call this vector as grad ϕ or $\nabla\phi$

$$\nabla\phi = \text{grad}\,\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \quad \text{—— (4)}$$

$$\frac{d\phi}{ds} = \nabla\phi \cdot \vec{u} \longrightarrow \text{directional derivative}$$

—— (5)